

Subrepresentations of Kronecker Representations

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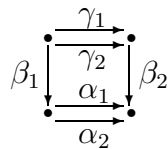
Abstract : Translated into the language of representations of quivers, a challenge in matrix pencil theory is to find sufficient and necessary conditions for a Kronecker representation to be a subfactor of another Kronecker representation in terms of their Kronecker invariants. The problem is reduced to a numerical criterion for a Kronecker representation to be a subrepresentation of another Kronecker representation in terms of their Kronecker invariants. The key to the problem is the calculation of ranks of matrices over polynomial rings. For this, a generalization and specialization approach is introduced. This approach is applied to provide a numerical criterion for a preprojective (resp. regular, preinjective) Kronecker representation to be a subrepresentation of another preprojective (resp. regular, preinjective) Kronecker representation in terms of their Kronecker invariants.

Introduction

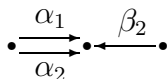
The classification of Kronecker representations was started by Weierstrass in 1867 and completed by Kronecker in 1890. A natural problem is to classify the subrepresentations of Kronecker representations, i.e., the pairs (N, M) in which N is a subrepresentation of a Kronecker representation M , just as has been done for uniserial rings by Ringel and Schmidmeier [9]. However this problem is hopeless to solve completely: Indeed,

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this problem is clearly equivalent to classifying those representations of the quiver



which satisfy the relations $\beta_2\gamma_1 - \alpha_1\beta_1 = \beta_2\gamma_2 - \alpha_2\beta_1 = 0$ and which are such that the maps β_1 and β_2 are inclusion maps [1]. For this, one would have to classify the representations of the quiver



which are such that the map β_2 is an inclusion map. This problem is clearly wild [8]. Nevertheless we may study the subrepresentations of Kronecker representations in another interesting way, namely, to find a numerical criterion for a Kronecker representation to be a subrepresentation of another Kronecker representation in terms of their Kronecker invariants. Later on we will see that the solution of this problem is also the solution of the first part of the challenge below.

Our original motivation is based on a challenge in matrix pencil theory. In [7, p. 329] the following question, which is closely related to pole placement, non-regular feedback, dynamic feedback, zero placement and early-stage design in control theory is declared to be a “challenge” by the authors.

Recall that a *matrix pencil* over a field \mathbb{K} is a matrix $\lambda E + H$ where λ is an indeterminate and E, H are matrices over \mathbb{K} of the same size. Two matrix pencils $\lambda E_1 + H_1$ and $\lambda E_2 + H_2$ of the same size are said to be *strictly equivalent*, denoted $\lambda E_1 + H_1 \sim \lambda E_2 + H_2$, if there exist invertible constant matrices P and Q such that $\lambda E_1 + H_1 = P(\lambda E_2 + H_2)Q$.

Challenge. [7] *Let $E, H \in \mathbb{R}^{(m+n) \times (p+q)}$ and $E', H' \in \mathbb{R}^{m \times p}$. Find necessary and sufficient conditions in terms of Kronecker invariants of the matrix pencils $\lambda E + H$ and $\lambda E' + H'$ for the existence of matrix pencils $F_{12}(\lambda)$, $F_{21}(\lambda)$ and $F_{22}(\lambda)$ such that $\lambda E + H \sim \begin{bmatrix} \lambda E' + H' & F_{12}(\lambda) \\ F_{21}(\lambda) & F_{22}(\lambda) \end{bmatrix}$ holds. Moreover, provide an algorithm for constructing $F_{12}(\lambda)$, $F_{21}(\lambda)$ and $F_{22}(\lambda)$ whenever a solution exists.*

The following was mentioned in [5, p. 62] : “The problem of giving necessary and sufficient conditions for the existence of a matrix pencil with prescribed Kronecker invariants and a prescribed arbitrary subpencil remains open and seems to be very difficult.” However, partial answers are known when $\lambda E + H$ and $\lambda E' + H'$ are both regular [2, 10, 11]; when $\lambda E + H$ is regular and $\lambda E' + H'$ is arbitrary [4]; when $\lambda E + H$ is arbitrary and $\lambda E' + H'$ is regular [5]; when $\lambda E + H$ has rank equal to the number of its rows and $\lambda E' + H'$ has rank equal to the number of its columns [3].

Three approaches, i.e., matrix pencil approach, polynomial approach, and geometric approach, have been used to attack the Challenge, see [7] and the references cited there. In this paper we provide the fourth approach, namely representations of quivers. Here we focus on the first part of the challenge.

The contents of this paper is organized as follows: In section 1, we first translate the Challenge into the language of representations of quivers. Thus the Challenge is found equivalent to finding sufficient and necessary conditions for a Kronecker representation to be a subfactor of another Kronecker representation in terms of their Kronecker invariants. Then the problem is reduced to finding a numerical criterion for a Kronecker representation to be a subrepresentation of another Kronecker representation in terms of their Kronecker invariants. And thus the problem becomes fairly elementary. The key point is to calculate the ranks of matrices over polynomial rings. Finally we extend the underlying field from the field of real numbers \mathbb{R} to the field of complex numbers \mathbb{C} and more generally we work on an arbitrary algebraically closed field \mathbb{K} . Thus the Kronecker invariants of a Kronecker representation can be expressed simply by a set of positive integers. In section 2, we consider the homomorphisms between two Kronecker representations, i.e., the matrix pairs that satisfy two equations [1]. We partition such a matrix pair into a block matrix pair. Via easy calculations one can learn the explicit form of each block in the matrix pair. This is very useful. In section 3, we obtain a numerical criterion for a preprojective (resp. regular, preinjective) Kronecker representation to be a subrepresentation of another preprojective (resp. regular, preinjective) Kronecker representation in terms of their Kronecker invariants. This follows from the calculation of the rank of matrices over polynomial rings using the generalization and specialization approach.

1 Reduction of the Challenge

1.1 Translation into the Language of Representations of Quivers

Recall that the *Kronecker quiver* is the quiver with two vertices 1, 2 and two arrows α and β from 1 to 2. A *Kronecker representation* M , i.e., a representation of the Kronecker quiver, can be written as $(M(1), M(2); M(\alpha), M(\beta))$ or $(M(\alpha), M(\beta))$, where $M(1), M(2)$ are the vector spaces associated with the vertices 1, 2 respectively and $M(\alpha), M(\beta) : M(1) \rightarrow M(2)$ are the linear maps that are represented by the arrows α and β , respectively. For more on representation theory of quivers we refer to [1]. Denote by \mathcal{K} the representation category of the Kronecker quiver. Note that in this paper we always consider subrepresentations up to isomorphism. As a result, we say a Kronecker representation $N = (N(\alpha), N(\beta))$ is a *subrepresentation* of a Kronecker representation $M = (M(\alpha), M(\beta))$ if there is a monomorphism from N to M , or

equivalently if there are injective linear maps ϕ and ψ such that $M(\alpha)\phi = \psi N(\alpha)$ and $M(\beta)\phi = \psi N(\beta)$. Dually, a Kronecker representation $N = (N(\alpha), N(\beta))$ is called a *factor representation* of a Kronecker representation $M = (M(\alpha), M(\beta))$ if there is an epimorphism from M to N , or equivalently if there are surjective linear maps ϕ and ψ such that $N(\alpha)\phi = \psi M(\alpha)$ and $N(\beta)\phi = \psi M(\beta)$. A *subfactor* of a Kronecker representation M is a factor representation of a subrepresentation of M , equivalently a subrepresentation of a factor representation of M .

Clearly a matrix pencil $\lambda E + H$ corresponds to a Kronecker representation (E, H) . Moreover, two matrix pencils $\lambda E_1 + H_1$ and $\lambda E_2 + H_2$ are strictly equivalent if and only if (E_1, H_1) and (E_2, H_2) are isomorphic as Kronecker representations, i.e., if there are invertible matrices G_1 and G_2 such that $G_2 E_1 = E_2 G_1$ and $G_2 H_1 = H_2 G_1$. In this way, the Challenge amounts to *finding matrices $E_{12}, E_{21}, E_{22}, H_{12}, H_{21}$, and H_{22} such that the two Kronecker representations (E, H) and $\left(\begin{bmatrix} E' & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} H' & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\right)$ are isomorphic*. If such a solution exists, then we write $\lambda E + H \succ \lambda E' + H'$ or $\lambda E' + H' \prec \lambda E + H$ or $(E, H) \succ (E', H')$ or $(E', H') \prec (E, H)$. Clearly, if $(E_1, H_1) \sim (E, H)$ and $(E, H) \succ (E', H')$, then $(E_1, H_1) \succ (E', H')$; and if $(E_2, H_2) \sim (E', H')$ and $(E, H) \succ (E', H')$, then $(E, H) \succ (E_2, H_2)$.

Proposition 1. *$(E, H) \succ (E', H')$ if and only if (E', H') is a subfactor of (E, H) . In particular, the relation \succ is a partial order on the set of all Kronecker representations.*

Proof. If $(E, H) \succ (E', H')$ then there are matrices $E_{12}, E_{21}, E_{22}, H_{12}, H_{21}$ and H_{22} such that two Kronecker representations (E, H) and $\left(\begin{bmatrix} E' & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} H' & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\right)$ are isomorphic. Since $\left(\begin{bmatrix} I \\ 0 \end{bmatrix}, I\right)$ is a monomorphism, $\left(\begin{bmatrix} E' \\ E_{21} \end{bmatrix}, \begin{bmatrix} H' \\ H_{21} \end{bmatrix}\right)$ is a subrepresentation of (E, H) . Furthermore, (E', H') is a factor representation of $\left(\begin{bmatrix} E' \\ E_{21} \end{bmatrix}, \begin{bmatrix} H' \\ H_{21} \end{bmatrix}\right)$, since $(I, [I \ 0])$ is an epimorphism. Thus (E', H') is a subfactor of (E, H) . Conversely, if (E', H') is a subfactor of (E, H) , then there is a subrepresentation (E_1, H_1) of (E, H) such that (E', H') is a factor representation of (E_1, H_1) . Hence, there are full rank matrices $A_i, B_i, i = 1, 2$, and A'_1, B'_2 such that $(E, H)A_2 = B_2(E_1, H_1)$, $(E', H')A_1 = B_1(E_1, H_1)$, $A_1 A'_1 = I$, and $B'_2 B_2 = I$. Since $B_1 B'_2$ and $A_2 A'_1$ are full rank matrices, there exist invertible matrices $C_i, D_i, i = 1, 2$, such that $B_1 B'_2 = C_1 [I \ 0] C_2$ and $A_2 A'_1 = D_1 \begin{bmatrix} I \\ 0 \end{bmatrix} D_2$. Consequently,

$$\begin{aligned} (E', H') &= (E', H')A_1 A'_1 = B_1(E_1, H_1)A'_1 = B_1 B'_2 B_2(E_1, H_1)A'_1 = B_1 B'_2 (E, H)A_2 A'_1 \\ &= C_1 [I \ 0] C_2 (E, H) D_1 \begin{bmatrix} I \\ 0 \end{bmatrix} D_2 \sim [I \ 0] C_2 (E, H) D_1 \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &\prec C_2 (E, H) D_1 \sim (E, H) . \end{aligned}$$
□

1.2 Reduction to the Subrepresentation Case

Once we find a sufficient and necessary condition $C(\mathcal{N}, \mathcal{M})$ for a Kronecker representation N to be a subrepresentation of another Kronecker representation M in terms of the Kronecker invariants \mathcal{N} and \mathcal{M} of N and M , then dually we will find a sufficient and necessary condition $C^*(\mathcal{M}, \mathcal{N})$ for N to be a factor representation of M . Furthermore, we will find a sufficient and necessary condition for N to be a subfactor of M : There exists a Kronecker module L of Kronecker invariants \mathcal{L} such that conditions $C(\mathcal{L}, \mathcal{M})$ and $C^*(\mathcal{L}, \mathcal{N})$ are satisfied. Therefore the question is reduced from the subfactor one to one of the subrepresentation.

Remark. The existence question in the condition is not very easy to handle, but it seems difficult to avoid. Indeed, existence question also appear in the results of [3, 4, 5].

1.3 Extension of the Underlying Field

Though the question is posed on the field of real numbers \mathbb{R} , we may consider the question on the field of complex numbers \mathbb{C} :

Proposition 2. *A real Kronecker representation (E', H') is a subrepresentation of another real Kronecker representation (E, H) over \mathbb{R} if and only if the same is the case over \mathbb{C} .*

Proof. The necessity is trivial. It remains to consider sufficiency. First, there are full column rank complex matrices P and Q such that $QE' = EP$ and $QH' = HP$. Second, let $P = P_1 + iP_2$ and $Q = Q_1 + iQ_2$ with $P_j, Q_j, j = 1, 2$, being real matrices and $i = \sqrt{-1}$. Then we have $Q_j E' = EP_j$ and $Q_j H' = HP_j$ for $j = 1, 2$. Since P (resp. Q) is of full column rank, $P_1 = P_2 = 0$ (resp. $Q_1 = Q_2 = 0$) can not occur. Hence $P_1 + xP_2$ (resp. $Q_1 + xQ_2$) is of smaller rank than P (resp. Q) for only finitely many values x in \mathbb{C} , i.e., the common roots of all rank P (resp. rank Q)–minors of $P_1 + xP_2$ (resp. $Q_1 + xQ_2$). Consequently there is some value x_0 in \mathbb{R} such that $P_1 + x_0P_2$ and $Q_1 + x_0Q_2$ are of full column rank, and $(P_1 + x_0P_2, Q_1 + x_0Q_2)$ is a monomorphism from (E', H') to (E, H) . \square

And more generally, we are able to consider the problem over an arbitrary algebraically closed field \mathbb{K} . By extension of the underlying field we can simply express Kronecker invariants as a set of integers (see section 2.1 below), this is of great benefit.

2 Homomorphisms between two Kronecker Representations

Note that a homomorphism between two Kronecker representations is just a pair of matrices satisfying two equations. In this section we partition these two matrices in

the natural way (corresponding to their direct sum decompositions of indecomposable representations) and observe the form of every block.

2.1 Kronecker Invariants

Denote by I the identity matrix and by J the Jordan block with eigenvalue 0 (of the appropriate size). Denote by $\mathbb{P}^1(\mathbb{K})$ the projective line over \mathbb{K} . By the well-known Krull-Schmidt theorem, a Kronecker representation can be decomposed into a direct sum of indecomposable Kronecker representations. Let $Q_i := (\mathbb{K}^{i-1}, \mathbb{K}^i; \begin{bmatrix} I & \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \\ & I \end{bmatrix})$, $R_{\infty, i} := (\mathbb{K}^i, \mathbb{K}^i; J, I)$, $R_{p, i} := (\mathbb{K}^i, \mathbb{K}^i; I, pI + J)$, and $J_i := (\mathbb{K}^i, \mathbb{K}^{i-1}; \begin{bmatrix} I & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ & 0 \end{bmatrix})$, $p \in \mathbb{K}$, $i \in \mathbb{N}_1 := \{1, 2, \dots\}$. Then the sets $\{Q_i | i \in \mathbb{N}_1\}$, $\{R_{p, i} | p \in \mathbb{P}^1(\mathbb{K}), i \in \mathbb{N}_1\}$ and $\{J_i | i \in \mathbb{N}_1\}$, called *preprojective*, *regular*, and *preinjective* indecomposable Kronecker representations respectively, constitute a complete set of nonisomorphic indecomposable Kronecker representations [1]. Up to isomorphism, a Kronecker representation M can be uniquely written as $M = (\oplus_{i=1}^{m^P} Q_{a_i}) \oplus (\oplus_{p \in \mathbb{P}^1(\mathbb{K})} \oplus_{i=1}^{m^p} R_{p, b_i^p}) \oplus (\oplus_{i=1}^{m^I} J_{c_i})$ for some positive integers $a_i, i = 1, \dots, m^P$; $b_i^p, i = 1, \dots, m^p, p \in \mathbb{P}^1(\mathbb{K})$; $c_i, i = 1, \dots, m^I$ (notice that the superscripts do not mean power). The Kronecker representation M is uniquely determined by a_i, b_i^p, c_i , which are called the *Kronecker invariants* of M . Moreover, a Kronecker representation is said to be *preprojective* (resp. *regular*, *preinjective*) if it is the direct sum of preprojective (resp. regular, preinjective) indecomposable representations.

Remark. Usually the Kronecker invariants of M viewed as a matrix pencil are referred to the *row minimal indices*, the *infinite elementary factors*, the *finite elementary factors*, and the *column minimal indices* [6, 7]. Over an algebraically closed field \mathbb{K} , they correspond to positive integers $a_i, b_i^\infty, b_i^p (p \in \mathbb{K}), c_i$ respectively.

2.2 Decomposition of Homomorphism

Let M and N be two Kronecker representations. Then $M = M^P \oplus M^R \oplus M^I$ and $N = N^P \oplus N^R \oplus N^I$ where $M^P = \oplus_{i=1}^{m^P} Q_{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_{m^P}$, $M^R = \oplus_{p \in \mathbb{P}^1(\mathbb{K})} \oplus_{i=1}^{m^p} R_{p, b_i^p}$ with $b_1^p \geq b_2^p \geq \dots \geq b_{m^p}^p$ for every $p \in \mathbb{P}^1(\mathbb{K})$, $M^I = \oplus_{i=1}^{m^I} J_{c_i}$ with $c_1 \geq c_2 \geq \dots \geq c_{m^I}$, $N^P = \oplus_{i=1}^{n^P} Q_{d_i}$ with $d_1 \geq d_2 \geq \dots \geq d_{n^P}$, $N^R = \oplus_{p \in \mathbb{P}^1(\mathbb{K})} \oplus_{i=1}^{n^p} R_{p, e_i^p}$ with $e_1^p \geq e_2^p \geq \dots \geq e_{n^p}^p$ for every $p \in \mathbb{P}^1(\mathbb{K})$, $N^I = \oplus_{i=1}^{n^I} J_{f_i}$ with $f_1 \geq f_2 \geq \dots \geq f_{n^I}$. Of course these numbers $a_i, b_i^p, c_i, d_i, e_i^p, f_i$ are positive integers. Once again the superscripts do not mean power here.

By [1; Theorem 7.5], any homomorphism of representations $\phi \in \text{Hom}_{\mathcal{K}}(N, M)$ can be written as $\phi = \begin{bmatrix} \phi^{PP} & 0 & 0 \\ \phi^{RP} & \phi^{RR} & 0 \\ \phi^{IP} & \phi^{IR} & \phi^{II} \end{bmatrix}$ where $\phi^{ST} \in \text{Hom}_{\mathcal{K}}(N^T, M^S)$ for $S, T \in \{P, R, I\}$.

Writing ϕ as a matrix pair, we have

$$\phi = (\phi^1, \phi^2) = \left(\begin{bmatrix} \phi^{PP1} & 0 & 0 \\ \phi^{RP1} & \phi^{RR1} & 0 \\ \phi^{IP1} & \phi^{IR1} & \phi^{II1} \end{bmatrix}, \begin{bmatrix} \phi^{PP2} & 0 & 0 \\ \phi^{RP2} & \phi^{RR2} & 0 \\ \phi^{IP2} & \phi^{IR2} & \phi^{II2} \end{bmatrix} \right)$$

with $\phi^{ST} = (\phi^{ST1}, \phi^{ST2})$ for $S, T \in \{P, R, I\}$.

2.3 Analysis of ϕ^{PP} and ϕ^{II}

We can write $M^P = (M^P(1), M^P(2); M^P(\alpha), M^P(\beta))$ and $N^P = (N^P(1), N^P(2); N^P(\alpha), N^P(\beta))$, where $M^P(1) = \mathbb{K}^{\sum_{i=1}^{m^P} (a_i - 1)}$, $M^P(2) = \mathbb{K}^{\sum_{i=1}^{m^P} a_i}$, $N^P(1) = \mathbb{K}^{\sum_{j=1}^{n^P} (d_j - 1)}$, $N^P(2) = \mathbb{K}^{\sum_{j=1}^{n^P} d_j}$, $M^P(\alpha)$ and $N^P(\alpha)$ are of the form $\text{diag} \left\{ \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \right\}$, and $M^P(\beta)$ and $N^P(\beta)$ are of the form $\text{diag} \left\{ \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \right\}$. We can write $\phi^{PP} = (\phi^{PP1}, \phi^{PP2})$ where ϕ^{PP1} and ϕ^{PP2} are $(\sum_{i=1}^{m^P} (a_i - 1)) \times (\sum_{j=1}^{n^P} (d_j - 1))$ and $(\sum_{i=1}^{m^P} a_i) \times (\sum_{j=1}^{n^P} d_j)$ matrices respectively. By partitioning into $m^P \times n^P$ block matrices in the natural way (corresponding to their direct sum decomposition), we have $\phi^{PP1} = (\phi_{ij}^{PP1})_{ij}$ and $\phi^{PP2} = (\phi_{ij}^{PP2})_{ij}$, $i = 1, \dots, m^P$, $j = 1, \dots, n^P$. Since $M^P(\alpha)\phi^{PP1} = \phi^{PP2}N^P(\alpha)$ and $M^P(\beta)\phi^{PP1} = \phi^{PP2}N^P(\beta)$, we have $\begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \phi_{ij}^{PP1} = \phi_{ij}^{PP2} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \phi_{ij}^{PP1} = \phi_{ij}^{PP2} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$. Therefore the blocks ϕ_{ij}^{PP1} and ϕ_{ij}^{PP2} have the form

$$\begin{bmatrix} x_1^{PPij} & & & \\ & \ddots & & \\ & & \ddots & \\ x_{a_i - d_j + 1}^{PPij} & & \ddots & x_1^{PPij} \\ & & \ddots & \ddots \\ & & & x_{a_i - d_j + 1}^{PPij} \end{bmatrix} \quad (1)$$

of size $(a_i - 1) \times (d_j - 1)$ and $a_i \times d_j$, respectively, in case $a_i \geq d_j$, and empty otherwise.

Similarly we can write $\phi^{II} = (\phi^{II1}, \phi^{II2})$ where ϕ^{II1} and ϕ^{II2} are $(\sum_{i=1}^{m^I} c_i) \times (\sum_{j=1}^{n^I} f_j)$ and $(\sum_{i=1}^{m^I} (c_i - 1)) \times (\sum_{j=1}^{n^I} (f_j - 1))$ matrices, respectively. We partition these into $m^I \times n^I$ block matrices in the natural way and have $\phi^{II1} = (\phi_{ij}^{II1})_{ij}$ and $\phi^{II2} = (\phi_{ij}^{II2})_{ij}$ where the blocks ϕ_{ij}^{II1} and ϕ_{ij}^{II2} have the form

$$\begin{bmatrix} x_1^{IIij} & \ddots & x_{f_j - c_i + 1}^{IIij} & & \\ & \ddots & \ddots & \ddots & \\ & & x_1^{IIij} & \ddots & x_{f_j - c_i + 1}^{IIij} \end{bmatrix} \quad (2)$$

of size $c_i \times f_j$ and $(c_i - 1) \times (f_j - 1)$, respectively, in case $c_i \leq f_j$, and empty otherwise.

2.4 Analysis of ϕ^{RR}

Note that we can write $M^R = (M^R(1), M^R(2); M^R(\alpha), M^R(\beta))$ and $N^R = (N^R(1), N^R(2); N^R(\alpha), N^R(\beta))$, where $M^R(1) = M^R(2) = \mathbb{K}^{\sum_{p \in \mathbb{P}^1(\mathbb{K})} \sum_{i=1}^{m^p} b_i^p}$, $N^R(1) = N^R(2) = \mathbb{K}^{\sum_{p \in \mathbb{P}^1(\mathbb{K})} \sum_{j=1}^{n^p} e_j^p}$, $M^R(\alpha)$ and $N^R(\alpha)$ are of the form $\text{diag}\{J, \dots, J, I, \dots, I\}$, and $M^R(\beta)$ and $N^R(\beta)$ are of the form $\text{diag}\{I, \dots, I, \dots, pI + J, \dots, pI + J, \dots\}$. We can write $\phi^{RR} = (\phi^{RR1}, \phi^{RR2})$ where ϕ^{RR1} and ϕ^{RR2} are $(\sum_{p \in \mathbb{P}^1(\mathbb{K})} \sum_{i=1}^{m^p} b_i^p) \times (\sum_{p \in \mathbb{P}^1(\mathbb{K})} \sum_{j=1}^{n^p} e_j^p)$ matrices. By [1; Theorem 7.5], we have $\phi^{RR1} = \text{diag}\{\phi^{RR1p}\}_{p \in \mathbb{P}^1(\mathbb{K})}$ and $\phi^{RR2} = \text{diag}\{\phi^{RR2p}\}_{p \in \mathbb{P}^1(\mathbb{K})}$. If we partition these matrices into $(\sum_{p \in \mathbb{P}^1(\mathbb{K})} m^p) \times (\sum_{p \in \mathbb{P}^1(\mathbb{K})} n^p)$ block matrices in the natural way, we have $\phi^{RR1p} = (\phi_{ij}^{RR1p})_{ij}$ and $\phi^{RR2p} = (\phi_{ij}^{RR2p})_{ij}$. Since $M^R(\alpha)\phi^{RR1} = \phi^{RR2}N^R(\alpha)$ and $M^R(\beta)\phi^{RR1} = \phi^{RR2}N^R(\beta)$, we have $J\phi_{ij}^{RR1\infty} = \phi_{ij}^{RR2\infty}J$ and $I\phi_{ij}^{RR1\infty} = \phi_{ij}^{RR2\infty}I$, $I\phi_{ij}^{RR1p} = \phi_{ij}^{RR2p}I$, and $(pI + J)\phi_{ij}^{RR1p} = \phi_{ij}^{RR2p}(pI + J)$ for every $p \in \mathbb{K}$. Therefore the block $\phi_{ij}^{RR1p} = \phi_{ij}^{RR2p}$ has the form

$$\begin{bmatrix} x_{g_{ij}^p}^{ppij} & \cdots & x_2^{ppij} & x_1^{ppij} \\ & \ddots & \ddots & x_2^{ppij} \\ & & \ddots & \ddots \\ & & & x_{g_{ij}^p}^{ppij} \end{bmatrix} \quad (3)$$

of size $b_i^p \times e_j^p$, where $g_{ij}^p = \min\{b_i^p, e_j^p\}$ for every $p \in \mathbb{P}^1(\mathbb{K})$ and $h \in \{1, 2\}$.

Remark. In a similar way, one can easily describe every block in ϕ^1 and ϕ^2 . However, as sections 2.3 and 2.4 are enough for later use, all other cases are omitted here.

3 Subrepresentations of Kronecker Representations

Assume that N' and M' are preprojective (resp. regular, or preinjective) Kronecker representations. In this section we provide a sufficient and necessary condition for N' to be a subrepresentation of M' in terms of their Kronecker invariants. For convenience, we consider the preprojective (resp. regular, or preinjective) parts of the Kronecker modules N and M given in section 2.2 instead of N' and M' .

A *generic matrix* is a matrix whose elements are pairwise different indeterminates. A matrix pair $\phi = (\phi^1, \phi^2)$ is called a *generic homomorphism* from N to M if ϕ^1 and ϕ^2 are generic matrices satisfying $M(\alpha)\phi^1 = \phi^2N(\alpha)$ and $M(\beta)\phi^1 = \phi^2N(\beta)$. Clearly, a generic homomorphism from N to M is a homomorphism from N to M over some transcendental extension field of \mathbb{K} . Once the indeterminates in the generic

homomorphism ϕ take special values in \mathbb{K} then ϕ becomes a homomorphism from N to M . Conversely, any homomorphism from N to M can be obtained in this way. From now on $\phi = (\phi^1, \phi^2)$ is always assumed to be a generic homomorphism from N to M . Clearly, N is a subrepresentation of M if and only if there exists a monomorphism from N to M , or if and only if the generic homomorphism $\phi = (\phi^1, \phi^2)$ from N to M is a monomorphism over some rational function field over \mathbb{K} , or if and only if ϕ^1 and ϕ^2 viewed as matrices over the polynomial rings, equivalently over their quotient fields, are of full column ranks. If we partition ϕ as done in section 2.2 and partition ϕ^{SSh} , $S \in \{P, R, I\}$, $h \in \{1, 2\}$, as done in section 2.3 and section 2.4, then the blocks in ϕ^{SSh} have the forms (1), (2) or (3) in sections 2.3 and 2.4 where all x_* are assumed to be indeterminates. Thus N^S is a subrepresentation of M^S if and only if ϕ^{SS1} and ϕ^{SS2} are of full column rank. In order to determine when ϕ^{SS1} and ϕ^{SS2} are of full column rank, we calculate the ranks of ϕ^{SS1} and ϕ^{SS2} .

3.1 Generalization and Specialization

In order to calculate the ranks of the matrices ϕ^{SS1} and ϕ^{SS2} , we employ the *generalization and specialization approach*. The *generalization* procedure consists of replacing some elements in the matrix of rational functions A with new independent indeterminates, so that the rank of the resulting matrix of rational functions provides an upper bound for the rank of the original matrix A . The *specialization* procedure consists of replacing some indeterminates in A with special values, usually 0 or 1, so that the rank of the resulting matrix provides a lower bound for the rank of the original matrix A . Usually, by a series elementary transformations of matrices and generalizations, we can obtain a matrix of rational functions B from A , and by specialization we can obtain a matrix C from A . It will be shown that $\text{rank } B = \text{rank } C$. Thus we conclude that $\text{rank } A = \text{rank } B = \text{rank } C$. In the following we will apply this approach to calculate the ranks of ϕ^{SS1} and ϕ^{SS2} .

First we calculate the ranks of block upper triangular generic matrices by the generalization-specialization approach. The rank formula obtained is closely related to the rank formula obtained in the preprojective-to-preprojective and preinjective-to-preinjective cases (see the remarks in section 3.3 and section 3.4 below).

Proposition 3. *Let $A = (A_{ij})_{ij}$ with $1 \leq i, j \leq q$ be a block upper triangular generic matrix, i.e., $A_{ij} = 0$ for $1 \leq j < i \leq q$ and A_{ij} is $r_i \times c_j$ generic matrix for $1 \leq i \leq j \leq q$. Assume that all indeterminates in A are different. Then $\text{rank } A = \min\{\sum_{j=1}^i r_j + \sum_{j=i+1}^q c_j \mid 0 \leq i \leq q\}$.*

Remark. By convention we require $\sum_{j=k}^i y_j = 0$ if $i < k$.

Proof. Let F be the transcendental extension field of \mathbb{K} obtained by adding all indeterminates in A , i.e., the field of rational functions in all indeterminates in A over

\mathbb{K} . We proceed by induction on q : It is trivial for case $q = 1$. Now consider the case $q \geq 2$.

If $r_1 \leq c_1$ then by elementary transformations over F , A can be reduced to another block upper triangular matrix $A' = (A'_{ij})_{ij}$, $1 \leq i, j \leq q$, where $A'_{ij} = 0$ for $1 \leq j < i \leq q$, $A'_{11} = [I, 0]$ and I is the $r_1 \times r_1$ identity matrix, and $A'_{1j} = 0$ for $2 \leq j \leq q$ with $A'_{ij} = A_{ij}$ for $2 \leq i \leq q$. By our induction hypothesis, $\text{rank } A = r_1 + \min\{\sum_{j=2}^i r_j + \sum_{j=i+1}^q c_j | 1 \leq i \leq q\} = \min\{\sum_{j=1}^i r_j + \sum_{j=i+1}^q c_j | 0 \leq i \leq q\}$.

If $r_1 > c_1$ then by elementary transformations over F , A can be reduced to another block upper triangular matrix $A' = (A'_{ij})_{ij}$, $1 \leq i, j \leq q$, where $A'_{ij} = 0$ for $1 \leq j < i \leq q$, $A'_{11} = I$ for the $c_1 \times c_1$ identity matrix, and $A'_{1j} = 0$ for $2 \leq j \leq q$ with $A'_{ij} = A_{ij}$ for $3 \leq i \leq q$. By generalization, i.e., replacing all elements in the $(r_1 + r_2 - c_1) \times c_j$ matrices A'_{2j} , $2 \leq j \leq q$, with different new indeterminates, we obtain a matrix B . By induction hypothesis, we get $\text{rank } A = \text{rank } A' \leq \text{rank } B = c_1 + \min\{\sum_{j=2}^q c_j, (r_1 + r_2 - c_1) + \sum_{j=3}^q c_j, (r_1 + r_2 - c_1) + r_3 + \sum_{j=4}^q c_j, \dots, (r_1 + r_2 - c_1) + \sum_{j=3}^q r_j\} = \min\{\sum_{j=1}^i r_j + \sum_{j=i+1}^q c_j | 0 \leq i \leq q\}$. On the other hand, by specialization, i.e., taking the $(1, 1), (2, 2), \dots, (c_1, c_1)$ entries of A to be 1 and all other indeterminates lying in the same rows or columns as these entries as 0. The resulting matrix C clearly has the same rank as B . Thus $\text{rank } A \geq \text{rank } C = \text{rank } B$. Finally $\text{rank } A = \text{rank } C = \text{rank } B = \min\{\sum_{j=1}^i r_j + \sum_{j=i+1}^q c_j | 0 \leq i \leq q\}$. \square

3.2 The Preprojective to Preprojective Case

Keeping in mind the analysis of ϕ^{PP} in section 2.3, let

$$\begin{aligned} r_1 &:= \max\{1 \leq j \leq n^P | d_j > a_1\}; \\ s_1 &:= \max\{1 \leq i \leq m^P | d_{r_1+1} \leq a_i\}; \\ &\dots \\ r_l &:= \max\{1 \leq j \leq n^P | d_j > a_{s_{l-1}+1}\}; \\ s_l &:= \max\{1 \leq i \leq m^P | d_{r_l+1} \leq a_i\}; \\ &\dots \\ r_t &= n^P. \end{aligned}$$

Note that r_1 is just the number of zero blocks in the first block row of ϕ^{PP2} , s_1 is just the number of the block rows of ϕ^{PP2} having the largest number of nonzero blocks. In the following all undefined numbers such as s_0 are assumed to be 0.

Proposition 4. $\text{rank } \phi^{PP2} = \min\{\sum_{j=1}^{s_i} a_j + \sum_{j=r_{i+1}+1}^{r_t} d_j | 0 \leq i \leq t-1\}$.

Proof. We calculate $\text{rank } \phi^{PP2}$ by induction on t . If $t = 1$ then $\phi^{PP2} = 0$ and we are done. Assume $t \geq 2$.

Case 1. $\sum_{j=1}^{s_i} a_j \geq \sum_{j=r_{i+1}+1}^{r_{i+1}+1} d_j, 1 \leq i \leq t-2$.

In this case we do not need to use induction. Clearly

$$\text{rank } \phi^{PP2} \leq \min\left\{\sum_{j=1}^{s_{t-1}} a_j, \sum_{j=r_1+1}^{r_t} d_j\right\} = \min\left\{\sum_{j=1}^{s_i} a_j + \sum_{j=r_{i+1}+1}^{r_t} d_j \mid 0 \leq i \leq t-1\right\}.$$

Next we prove that $\text{rank } \phi^{PP2} \geq \min\left\{\sum_{j=1}^{s_{t-1}} a_j, \sum_{j=r_1+1}^{r_t} d_j\right\}$. We proceed by specialization, namely we let the indeterminates in ϕ^{PP2} take the special values 0 or 1 such that the resulting matrix is of rank $\min\left\{\sum_{j=1}^{s_{t-1}} a_j, \sum_{j=r_1+1}^{r_t} d_j\right\}$.

(1) If $a_1 \geq \sum_{j=r_1+1}^{r_t} d_j$ then let the $(1, \sum_{j=1}^{r_1} d_j + 1), (2, \sum_{j=1}^{r_1} d_j + 2), \dots, (\sum_{j=r_1+1}^{r_t} d_j, \sum_{j=1}^{r_t} d_j)$ elements of ϕ^{PP2} take 1, and let all other indeterminates take 0. This finishes the specialization.

(2) If $\sum_{j=r_1+1}^{u_1} d_j \leq a_1 < \sum_{j=r_1+1}^{u_1+1} d_j$ for some $r_1 + 1 \leq u_1 < r_t$ then we let the $(1, \sum_{j=1}^{u_1} d_j + 1), (2, \sum_{j=1}^{u_1} d_j + 2), \dots, (a_1 - \sum_{j=r_1+1}^{u_1} d_j, a_1 + \sum_{j=1}^{r_1} d_j), (a_1 - \sum_{j=r_1+1}^{u_1} d_j + 1, \sum_{j=1}^{r_1} d_j + 1), (a_1 - \sum_{j=r_1+1}^{u_1} d_j + 2, \sum_{j=1}^{r_1} d_j + 2), \dots, (a_1, \sum_{j=1}^{u_1} d_j)$ entries of ϕ^{PP2} take the value 1. If $t = 2$ then all other indeterminates are set to 0. This ends the specialization.

(3) For $t \geq 3$ suppose $a_2 < d_{u_1+1}$. Then $s_1 = 1$ and $r_2 \geq u_1 + 1$. This contradicts the assumption $\sum_{j=1}^{s_1} a_j \geq \sum_{j=r_1+1}^{r_2} d_j$. Thus $a_2 \geq d_{u_1+1}$. If $a_1 + a_2 \geq \sum_{j=r_1+1}^{r_t} d_j$ then we set the $(a_1 + a_2 - (\sum_{j=r_1+1}^{u_1+1} d_j - a_1) + 1, \sum_{j=1}^{r_1} d_j + a_1 + 1), (a_1 + a_2 - (\sum_{j=r_1+1}^{u_1+1} d_j - a_1) + 2, \sum_{j=1}^{r_1} d_j + a_1 + 2), \dots, (a_1 + a_2, \sum_{j=1}^{u_1+1} d_j); (a_1 + 1, \sum_{j=1}^{u_1+1} d_j + 1), (a_1 + 2, \sum_{j=1}^{u_1+1} d_j + 2), \dots, (a_1 + \sum_{j=u_1+2}^{r_t} d_j, \sum_{j=1}^{r_t} d_j)$ entries of ϕ^{PP2} equal to 1, and choose all other indeterminates as 0.

(4) If $\sum_{j=r_1+1}^{u_2} d_j \leq a_1 + a_2 < \sum_{j=r_1+1}^{u_2+1} d_j$ for some $r_1 + 1 \leq u_2 < r_t$, then we set the $(a_1 + a_2 - (\sum_{j=r_1+1}^{u_1+1} d_j - a_1) + 1, \sum_{j=1}^{r_1} d_j + a_1 + 1), (a_1 + a_2 - (\sum_{j=r_1+1}^{u_1+1} d_j - a_1) + 2, \sum_{j=1}^{r_1} d_j + a_1 + 2), \dots, (a_1 + a_2, \sum_{j=1}^{u_1+1} d_j); (a_1 + 1, \sum_{j=1}^{u_2} d_j + 1), (a_1 + 2, \sum_{j=1}^{u_2} d_j + 2), \dots, (a_1 + (a_1 + a_2 - \sum_{j=r_1+1}^{u_2} d_j), \sum_{j=1}^{r_1} d_j + a_1 + a_2); (a_1 + (a_1 + a_2 - \sum_{j=r_1+1}^{u_2} d_j) + 1, \sum_{j=1}^{u_1+1} d_j + 1), (a_1 + (a_1 + a_2 - \sum_{j=r_1+1}^{u_2} d_j) + 2, \sum_{j=1}^{u_1+1} d_j + 2), \dots, (a_1 + a_2 - (\sum_{j=r_1+1}^{u_1+1} d_j - a_1), \sum_{j=1}^{u_2} d_j)$ elements of ϕ^{PP2} equal to 1. If $t = 3$ we set all other indeterminates equal to 0.

(5) For $t \geq 4$ suppose $a_3 < d_{u_2+1}$. Then there exists some $s_i = 2$ with $1 \leq i \leq 2$ such that $r_{i+1} \geq u_2 + 1$. This contradicts the assumption $\sum_{j=1}^{s_i} a_j \geq \sum_{j=r_i+1}^{r_{i+1}} d_j$. Thus $a_3 \geq d_{u_2+1}$. If $a_1 + a_2 + a_3 \geq \sum_{j=r_1+1}^{r_t} d_j$, then we let the $(a_1 + a_2 + a_3 - (\sum_{j=r_1+1}^{u_2+1} d_j - a_1 - a_2) + 1, \sum_{j=1}^{r_1} d_j + a_1 + a_2 + 1), (a_1 + a_2 + a_3 - (\sum_{j=r_1+1}^{u_2+1} d_j - a_1 - a_2) + 2, \sum_{j=1}^{r_1} d_j + a_1 + a_2 + 2), \dots, (a_1 + a_2 + a_3, \sum_{j=1}^{u_2+1} d_j), (a_1 + a_2 + 1, \sum_{j=1}^{u_2+1} d_j + 1); (a_1 + a_2 + 2, \sum_{j=1}^{u_2+1} d_j + 2), \dots, (a_1 + a_2 + \sum_{j=u_2+2}^{r_t} d_j, \sum_{j=1}^{r_t} d_j)$ elements of ϕ^{PP2} be 1, and set all other indeterminates to be 0.

Proceeding in this way, this process will end with one of two possibilities:

- (i) we can proceed in $2s_{t-1}$ steps: In this case all nonzero rows are exhausted.
- (ii) we can proceed in $2q - 1$ steps with $1 \leq q \leq s_{t-1}$: In this case all nonzero columns are exhausted.

Via our specializations we have obtained a $(0,1)$ -matrix whose rank is $\sum_{j=1}^{s_{t-1}} a_j$ (resp. $\sum_{j=r_1+1}^{r_t} d_j$) in the case (i) (resp. (ii)): Indeed this $(0,1)$ -matrix can be reduced by elementary transformations to a $(0,1)$ -matrix for which in case (i) (resp. (ii)) there are just $\sum_{j=1}^{s_{t-1}} a_j$ (resp. $\sum_{j=r_1+1}^{r_t} d_j$) elements 1 lying in different rows and columns (by keeping the ones as far to the left as possible).

Case 2. Assume that $\sum_{j=1}^{s_i} a_j < \sum_{j=r_1+1}^{r_{i+1}} d_j$ for some $1 \leq i \leq t-2$ and let $v := \min\{1 \leq i \leq t-2 \mid \sum_{j=1}^{s_i} a_j < \sum_{j=r_1+1}^{r_{i+1}} d_j\}$. Let $A := \phi^{PP2}(1, \dots, \sum_{j=1}^{s_v} a_j; \sum_{j=1}^{r_1} d_j + 1, \dots, \sum_{j=1}^{r_{v+1}} d_j)$ be the submatrix of ϕ^{PP2} which is the intersection of the 1-st, ..., $(\sum_{j=1}^{s_v} a_j)$ -th rows of ϕ^{PP2} and the $(\sum_{j=1}^{r_1} d_j + 1)$ -st, ..., $(\sum_{j=1}^{r_{v+1}} d_j)$ -th columns of ϕ^{PP2} . By case 1, we have $\text{rank } A = \sum_{j=1}^{s_v} a_j$. By the induction hypothesis, the rank of the submatrix $B := \phi^{PP2}(\sum_{j=1}^{s_v} a_j + 1, \dots, \sum_{j=1}^{s_{t-1}} a_j; \sum_{j=1}^{r_{v+1}} d_j + 1, \dots, \sum_{j=1}^{r_t} d_j)$ of ϕ^{PP2} is equal to $\min\{\sum_{j=s_v+1}^{s_i} a_j + \sum_{j=r_{i+1}+1}^{r_t} d_j \mid v \leq i \leq t-1\}$. Thus $\text{rank } \phi^{PP2} = \text{rank } A + \text{rank } B = \sum_{j=1}^{s_v} a_j + \min\{\sum_{j=s_v+1}^{s_i} a_j + \sum_{j=r_{i+1}+1}^{r_t} d_j \mid v \leq i \leq t-1\} = \min\{\sum_{j=1}^{s_i} a_j + \sum_{j=r_{i+1}+1}^{r_t} d_j \mid 0 \leq i \leq t-1\}$. \square

Note that $r_1 = \max\{1 \leq j \leq n^P \mid d_j - 1 > a_1 - 1\}$; $s_1 = \max\{1 \leq i \leq m^P \mid d_{r_1+1} - 1 \leq a_i - 1\}$; ... ; $r_l = \max\{1 \leq j \leq n^P \mid d_j - 1 > a_{s_{l-1}+1} - 1\}$; $s_l = \max\{1 \leq i \leq m^P \mid d_{r_l+1} - 1 \leq a_i - 1\}$; ... ; $r_t = n^P$. By Proposition 4 we have the following formula on $\text{rank } \phi^{PP1}$.

Corollary 5. $\text{rank } \phi^{PP1} = \min\{\sum_{j=1}^{s_i} (a_j - 1) + \sum_{j=r_{i+1}+1}^{r_t} (d_j - 1) \mid 0 \leq i \leq t-1\}$.

Remark. By Proposition 4, Corollary 5, and Proposition 3 we find that $\text{rank } \phi^{PPh}$, $h \in \{1, 2\}$, is equal to the rank of the matrix obtained from ϕ^{PPh} by replacing each nonzero block in ϕ^{PPh} with a generic matrix of the same size. (Of course all indeterminates in these generic matrices are assumed to be different.)

By Proposition 4 and Corollary 5 we obtain a numerical criterion for a preprojective Kronecker representation to be a subrepresentation of another preprojective Kronecker representation in terms of their Kronecker invariants.

Theorem 6. N^P is a subrepresentation of M^P if and only if $r_1 = 0$, $\sum_{j=1}^{s_i} a_j \geq \sum_{j=r_1+1}^{r_{i+1}} d_j$ and $\sum_{j=1}^{s_i} (a_j - 1) \geq \sum_{j=r_1+1}^{r_{i+1}} (d_j - 1)$, $1 \leq i \leq t-1$.

3.3 The Preinjective to Preinjective Case

Keep in mind the analysis of ϕ^I in section 2.3. Let

$$\begin{aligned} u_1 &:= \max\{1 \leq j \leq m^I \mid c_j > f_1\}; \\ v_1 &:= \max\{1 \leq i \leq n^I \mid c_{u_1+1} \leq f_i\}; \\ &\dots \\ u_l &:= \max\{1 \leq j \leq m^I \mid c_j > f_{v_{l-1}+1}\}; \\ v_l &:= \max\{1 \leq i \leq n^I \mid c_{u_l+1} \leq f_i\}; \\ &\dots \\ u_w &= m^I. \end{aligned}$$

Note that u_1 is just the number of the zero blocks in the first block column of ϕ^{II1} , v_1 is just the number of the block columns of ϕ^{II1} having the largest number of nonzero blocks. Dual to Proposition 4 and Corollary 5 we have:

Proposition 7. $\text{rank } \phi^{II1} = \min\{\sum_{j=1}^{v_i} f_j + \sum_{j=u_{i+1}+1}^{u_w} c_j \mid 0 \leq i \leq w-1\}.$

Corollary 8. $\text{rank } \phi^{II2} = \min\{\sum_{j=1}^{v_i} (f_j - 1) + \sum_{j=u_{i+1}+1}^{u_w} (c_j - 1) \mid 0 \leq i \leq w-1\}.$

Remark. By Proposition 7, Corollary 8, and Proposition 3 we find that $\text{rank } \phi^{IIh}, h \in \{1, 2\}$, is equal to the rank of the matrix obtained from ϕ^{IIh} by replacing each nonzero block in ϕ^{IIh} with a generic matrix of the same size.

Theorem 9. N^I is a subrepresentation of M^I if and only if $v_{w-1} = v_w = n^I, \sum_{j=v_i+1}^{v_{w-1}} f_j \leq \sum_{j=u_{i+1}+1}^{u_w} c_j$ and $\sum_{j=v_i+1}^{v_{w-1}} (f_j - 1) \leq \sum_{j=u_{i+1}+1}^{u_w} (c_j - 1), 0 \leq i \leq w-2.$

3.4 The Regular to Regular Case

This case is easier.

Proposition 10. $\text{rank } \phi^{RRh} = \sum_{p \in \mathbb{P}^1(\mathbb{K})} \sum_{i=1}^{\min\{m^p, n^p\}} \min\{b_i^p, e_i^p\}, h \in \{1, 2\}.$

Proof. Keep in mind the analysis of ϕ^{RR} in section 2.4. For every $p \in \mathbb{P}^1(\mathbb{K})$ and every $h \in \{1, 2\}$ we keep the first nonzero element in each row of the matrices ϕ_{11}^{RRhp} and use it to eliminate all other entries in ϕ^{RRhp} which lie in the same row or column by elementary transformations over the transcendental extension field F of \mathbb{K} obtained by adding all indeterminates in ϕ^{RRh} . Next keep the first nonzero element in each row of the matrices ϕ_{22}^{RRhp} , and use them to eliminate all other elements in ϕ^{RRhp} which lie in the same row or column by elementary transformations over F . Proceeding in this way, after $\min\{m^p, n^p\}$ steps, we obtain $\sum_{i=1}^{\min\{m^p, n^p\}} \min\{b_i^p, e_i^p\}$ nonzero elements which lie in different rows and different columns of ϕ^{RRhp} , while all other entries in ϕ^{RRhp} are reduced to 0. Thus $\text{rank } \phi^{RRhp} = \sum_{i=1}^{\min\{m^p, n^p\}} \min\{b_i^p, e_i^p\}$. Furthermore $\text{rank } \phi^{RRh} = \sum_{p \in \mathbb{P}^1(\mathbb{K})} \sum_{i=1}^{\min\{m^p, n^p\}} \min\{b_i^p, e_i^p\}, h \in \{1, 2\}.$ \square

Theorem 11. N^R is a subrepresentation of M^R if and only if $m^p \geq n^p$ and $b_i^p \geq e_i^p, p \in \mathbb{P}^1(\mathbb{K}), 1 \leq i \leq n^p.$

Remark. In the same way, one can show that N^P is a subrepresentation of M^I if and only if $\sum_{i=1}^{m^I} (c_i - 1) \geq \sum_{i=1}^{n^P} d_i$. However, to solve the problem completely, i.e., for arbitrary Kronecker representations N and M , more analysis is needed.

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